Solutions, 316-XXIII

April 19, 2003

23(4/15) Complex eigenvalues
9.6[1,2*,5*,6,13(a*,b,c,d)]
CAUTION: there may be errors!!!

1 Problem 9.6.1

Find a general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -4 \\ 2 & -2 \end{pmatrix} .$$

Solution:
Form the characteristic polynomial:

$$\begin{aligned} P(r) &:= \det(\mathbf{A} - r\mathbf{I}) = \det \begin{pmatrix} 2-r & -4 \\ 2 & -2-r \end{pmatrix} \\ &= (r+2)(r-2) + 8 = r^2 + 4 = (r+2i)(r-2i) = 0 \end{aligned}$$

so the problem has complex eigenvalues/vectors. We work with the eigenvalue $r_1 = +2i$; the results for $r_2 = -2i$ are found by *complex conjugation* of the answer (i.e. change the sign of i everywhere). We have

$$\begin{aligned} \begin{pmatrix} -2-2i & -4 \\ 2 & -2-2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2 \\ -1-i \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned}$$

We know from the theory that when the matrix has a complex eigenvalue $r = \alpha \pm i\beta$ with eigenvector $\mathbf{u} = \mathbf{a} \pm i\mathbf{b}$ then the two linearly independent solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are $\mathbf{x}_1 = e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b}$, and $\mathbf{x}_2 = e^{\alpha t}\sin\beta t\mathbf{a} + e^{\alpha t}\cos\beta t\mathbf{b}$. Here this gives, with $r = 0 + 2i$:

$$\mathbf{x}_1 = \cos 2t \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and

$$\mathbf{x}_2 = \sin 2t \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

so that the general solution is given by

$$\mathbf{x}_{gen}(t) = C_1 \begin{pmatrix} 2\cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} + C_2 \begin{pmatrix} 2\sin 2t \\ -\sin 2t - \cos 2t \end{pmatrix} .$$

2 Problem 9.6.2

Find a general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the matrix

$$\mathbf{A} = \begin{pmatrix} -2 & -2 \\ 4 & 2 \end{pmatrix} .$$

Solution:
Form the characteristic polynomial:

$$\begin{aligned} P(r) &:= \det(\mathbf{A} - r\mathbf{I}) = \det \begin{pmatrix} -2-r & -2 \\ 4 & 2-r \end{pmatrix} \\ &= (r+2)(r-2) + 8 = r^2 + 4 = (r+2i)(r-2i) = 0 \end{aligned}$$

so the problem has complex eigenvalues/vectors. We work with the eigenvalue $r_1 = +2i$; the results for $r_2 = -2i$ are found by *complex conjugation* of the answer (i.e. change the sign of i everywhere). We have

$$\begin{aligned} \begin{pmatrix} -2-2i & -2 \\ 4 & 2-2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 \\ -1-i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned}$$

We know from the theory that when the matrix has a complex eigenvalue $r = \alpha \pm i\beta$ with eigenvector $\mathbf{u} = \mathbf{a} \pm i\mathbf{b}$ then the two linearly independent solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are $\mathbf{x}_1 = e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b}$, and $\mathbf{x}_2 = e^{\alpha t}\sin\beta t\mathbf{a} + e^{\alpha t}\cos\beta t\mathbf{b}$. Here this gives, with $r = 0 + 2i$:

$$\mathbf{x}_1 = \cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and

$$\mathbf{x}_2 = \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \cos 2t \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

so that the general solution is given by

$$\mathbf{x}_{gen}(t) = C_1 \begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} + C_2 \begin{pmatrix} \sin 2t \\ -\sin 2t - \cos 2t \end{pmatrix} .$$

3 Problem 9.6.5

Find a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 8 & -1 \end{pmatrix} .$$

Solution:
Form the characteristic polynomial:

$$\begin{aligned} P(r) &:= \det(\mathbf{A} - r\mathbf{I}) = \det \begin{pmatrix} -1-r & -2 \\ 8 & -1-r \end{pmatrix} \\ &= (r+1)^2 + 16 = (r-1-4i)(r-1+4i) = 0 \end{aligned}$$

so the problem has complex eigenvalues/vectors. We work with the eigenvalue $r_1 = -1 + 4i$; the results for $r_2 = -1 - 4i$ are found by *complex conjugation* of the answer (i.e. change the sign of i everywhere). We have

$$\begin{aligned} \begin{pmatrix} -4i & -2 \\ 8 & -4i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 \\ -2i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -2 \end{pmatrix} \end{aligned}$$

We know from the theory that when the matrix has a complex eigenvalue $r = \alpha \pm i\beta$ with eigenvector $\mathbf{u} = \mathbf{a} \pm i\mathbf{b}$ then the two linearly independent solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are $\mathbf{x}_1 = e^{\alpha t} \cos\beta t \mathbf{a} - e^{\alpha t} \sin\beta t \mathbf{b}$, and $\mathbf{x}_2 = e^{\alpha t} \sin\beta t \mathbf{a} + e^{\alpha t} \cos\beta t \mathbf{b}$. Here this gives, with $r = -1 + 4i$:

$$\mathbf{x}_1 = \cos 2t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

and

$$\mathbf{x}_2 = \sin 2t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos 2t \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

so that the Fundamental Matrix is given by

$$\mathbf{\Phi}(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ 2\sin 2t & -2\cos 2t \end{pmatrix} .$$

4 Problem 9.6.13a

Find the solution to the IVP:

$$\mathbf{x}' = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{x} \ , \ \mathbf{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \mathbf{x} \ .$$

Solution:
Form the characteristic polynomial:

$$\begin{aligned} P(r) &:= \det(\mathbf{A} - r\mathbf{I}) = \det \begin{pmatrix} -3-r & -1 \\ 2 & -1-r \end{pmatrix} \\ &= (r+1)(r+3) + 2 = r^2 + 4r + 5 = (r+2)^2 + 1 = 0 \end{aligned}$$

so the problem has complex eigenvalues/vectors. We work with the eigenvalue $r_1 = -2 + i$; the eigenvector is found from

$$\begin{aligned} \begin{pmatrix} -3+2-i & -1 \\ 2 & -1+2-i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 \\ -1-i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned}$$

where the first equation was used to find the components of the eigenvector. Following problem (1), we find an independent set as (with $r = -2 + i$):

$$\mathbf{x}_1 = e^{-2t} \cos t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - e^{-2t} \sin t \begin{pmatrix} 0 \\ -1 \end{pmatrix} = e^{-2t} \begin{pmatrix} \cos t \\ \sin t - \cos t \end{pmatrix}$$

and

$$\mathbf{x}_2 = e^{-2t} \sin t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-2t} \cos t \begin{pmatrix} 0 \\ -1 \end{pmatrix} = e^{-2t} \begin{pmatrix} \sin t \\ -\cos t - \sin t \end{pmatrix}$$

so that the general solution is given by

$$\mathbf{x}_{gen}(t) = C_1 e^{-2t} \begin{pmatrix} \cos t \\ \sin t - \cos t \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} \sin t \\ -\cos t - \sin t \end{pmatrix} \ .$$

A fundamental matrix for the system is then

$$\Phi(t) = e^{-2t} \begin{pmatrix} \cos t & \sin t \\ \sin t - \cos t & -\cos t - \sin t \end{pmatrix} \ ,$$

and the general solution is given in terms of the Fundamental matrix by the equivalent form

$$\mathbf{x}_{gen}(t) = \Phi(t) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Solving for the IC:

$$\begin{aligned} \Phi((0)) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \frac{1}{-1} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{aligned}$$

Finally, the solution to the IVP is:

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= (\mathbf{x}_2(t) - \mathbf{x}_1(t)) \\ &= e^t \begin{pmatrix} -\cos t + \sin t \\ -2\sin t \end{pmatrix} . \end{aligned}$$

5 Problem 9.6.13d modified

Find the solution to the IVP:

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 5 & -1 \end{pmatrix} \mathbf{x} \ , \ \mathbf{x}(\pi/2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ .$$

Solution:
Form the characteristic polynomial:

$$\begin{aligned} P(r) &:= \det(\mathbf{A} - r\mathbf{I}) = \det \begin{pmatrix} 3-r & -1 \\ 5 & -1-r \end{pmatrix} \\ &= (r+1)(r-3) + 3 = r^2 - 2r + 2 = (r-1)^2 + 1 = 0 \end{aligned}$$

so the problem has complex eigenvalues/vectors. We work with the eigenvalue $r_1 = 1 + i$; the eigenvector is found from

$$\begin{aligned} \begin{pmatrix} 3-1-i & -1 \\ 5 & -1-1-i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2+i \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

where the second equation was used to find the components of the eigenvector. Following problem (1), we find an independent set as (with $r = 1+i$):

$$\mathbf{x}_1 = e^t \cos t \begin{pmatrix} 2 \\ 5 \end{pmatrix} - e^t \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^t \begin{pmatrix} 2\cos t - \sin t \\ 5\cos t \end{pmatrix}$$

and

$$\mathbf{x}_2 = e^t \sin t \begin{pmatrix} 2 \\ 5 \end{pmatrix} + e^t \cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^t \begin{pmatrix} \cos t + 2\sin t \\ 5\sin t \end{pmatrix}$$

so that the general solution is given by

$$\mathbf{x}_{gen}(t) = C_1 e^t \begin{pmatrix} 2\cos t - \sin t \\ 5\cos t \end{pmatrix} + C_2 e^t \begin{pmatrix} \cos t + 2\sin t \\ 5\sin t \end{pmatrix} \ .$$

A fundamental matrix for the system is then

$$\Phi(t) = e^t \begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ 5\cos t & 5\sin t \end{pmatrix} \ ,$$

and the general solution is given in terms of the Fundamental matrix by the equivalent form

$$\mathbf{x}_{gen}(t) = \Phi(t)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Solving for the IC:

$$\begin{aligned}
\Phi((\pi/2))\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
e^{\pi/2}\begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \frac{e^{-\pi/2}}{-5}\begin{pmatrix} 5 & -2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{e^{-\pi/2}}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}
\end{aligned}$$

Finally, the solution to the IVP is:

$$\begin{aligned}
\mathbf{x}(t) &= \frac{e^{-\pi/2}}{5}\Phi(t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
&= \frac{e^{-\pi/2}}{5}\left(2\mathbf{x}_1(t) + \mathbf{x}_2(t)\right) \\
&= e^{t-\pi/2}\begin{pmatrix} \cos t \\ \sin t + 2\cos t \end{pmatrix} .
\end{aligned}$$