Solutions, 316-XXII

April 16, 2003

22(4/10) Eigenvalues, eigenvectors; Real eigenvalues
9.4[1,3*,5,7,11,19*,21*], 9.5[1,3,7*,11*,13,17*]
CAUTION: there may be errors!!!

1 Problem 9.4.3

Write the given system in the matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$.

$$\begin{aligned}
\frac{dx}{dt} &= t^2 x - y - z + t \ , \\
\frac{dy}{dt} &= e^t z + 5 \ , \\
\frac{dz}{dt} &= tx - y + 3z - e^t \ .
\end{aligned}$$

Solution:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t^2 & -1 & -1 \\ 0 & 0 & e^t \\ t & -1 & 3 \end{pmatrix} + \begin{pmatrix} t \\ 5 \\ -e^t \end{pmatrix}$$

2 Problem 9.4.19

The given vector functions are solutions of the system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$; determine whether they form a fundamental set. If they do, find a fundamental matrix and give the general solution.

$$\mathbf{x}_1 = e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \ , \ \mathbf{x}_2 = e^{2t} \begin{pmatrix} -2 \\ 4 \end{pmatrix} \ .$$

Solution:
We form the matrix

$$\mathbf{\Phi}(t) = e^{2t} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \ .$$

Since $\det \mathbf{\Phi}(t) = e^{4t}(4-4) = 0$ the two functions do not form a fundamental set, and we cannot write a fundamental solution matrix.

3 Problem 9.4.20

The given vector functions are solutions of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}+\mathbf{f}$; determine whether they form a fundamental set. If they do, find a fundamental matrix and give the general solution.

$$\mathbf{x}_1 = e^{-t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} \ , \ \mathbf{x}_2 = e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \ .$$

Solution:
We form the matrix

$$\mathbf{\Phi}(t) = \begin{pmatrix} 3e^{-t} & e^{4t} \\ 2e^{-t} & -e^{4t} \end{pmatrix} \ .$$

Since $\det \mathbf{\Phi}(t) = e^{3t}(-3-2) = -5e^{3t} \neq 0$ the two functions form a fundamental set, and $\mathbf{\Phi}(t)$ is a fundamental solution matrix. The general solution is then

$$\mathbf{x}_{gen}(t) = \mathbf{\Phi}(t)\mathbf{c}$$

with $\mathbf{c}$ an arbitrary constant vector.

4 Problem 9.4.21

The given vector functions are solutions of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}+\mathbf{f}$; determine whether they form a fundamental set. If they do, find a fundamental matrix and give the general solution.

$$\mathbf{x}_1 = \begin{pmatrix} e^{-t} \\ 2e^{-t} \\ e^{-t} \end{pmatrix} \ , \ \mathbf{x}_2 = e^t \begin{pmatrix} e^{t} \\ 0 \\ e^{t} \end{pmatrix} \ , \ \mathbf{x}_3 = \begin{pmatrix} e^{3t} \\ -e^{3t} \\ 2e^{3t} \end{pmatrix} \ .$$

Solution:
We form the matrix

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{-t} & e^{t} & e^{3t} \\ 2e^{-t} & 0 & -e^{3t} \\ e^{-t} & e^{t} & 2e^{3t} \end{pmatrix} \ .$$

Now, compute the determinant:

$$\begin{aligned} \det \mathbf{\Phi}(t) &= e^{-t} \begin{pmatrix} 0 & -e^{3t} \\ e^{t} & 2e^{3t} \end{pmatrix} - 2e^{-t} \begin{pmatrix} e^{t} & e^{3t} \\ e^{t} & 2e^{3t} \end{pmatrix} + e^{-t} \begin{pmatrix} e^{t} & e^{3t} \\ 0 & -e^{3t} \end{pmatrix} \ , \\ &= e^{-t}e^{4t} - 2e^{-t}e^{4t} - e^{-t}e^{4t} = -2e^{3t} \neq 0 \ . \end{aligned}$$

Since the determinant is non-zero, $\mathbf{\Phi}(t)$ is a fundamental matrix and the general solution can be written as

$$\mathbf{x}_{gen}(t) = \mathbf{\Phi}(t)\mathbf{c}$$

with $\mathbf{c}$ and arbitrary constan vector.

5 Problem 9.5.7

Find the eigenvalues and eigenvectors of the given matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix} .$$

Solution:

Find the eigenvalues and eigenvectors of the given matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix} .$$

Solution:

Form the characteristic polynomial:

$$\begin{aligned} P(r) &:= \det(\mathbf{A} - r\mathbf{I}) = \det \begin{pmatrix} 1-r & 0 & 0 \\ 2 & 3-r & 1 \\ 0 & 2 & 4-r \end{pmatrix} \\ &= (1-r)\left((3-r)(4-r)-2\right) = (1-r)(r^2-7r+10) \\ &= (1-r)(r-2)(r-5) \end{aligned}$$

The eigenvectors are found from the definition:

$$\mathbf{A}\mathbf{v}_i = r_i \mathbf{v}_i \ :$$

or

$$\begin{pmatrix} 1-r & 0 & 0 \\ 2 & 3-r & 1 \\ 0 & 2 & 4-r \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

1. Eigenvalue $r_1 = 5$:

$$\begin{pmatrix} -4 & 0 & 0 \\ 2 & -2 & 1 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

2. Eigenvalue $r_2 = 2$:

$$\begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

3. Eigenvalue $r_3 = 1$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

6 Problem 9.5.6

Find the eigenvalues and eigenvectors of the given matrix:

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 4 & -8 & 2 \end{pmatrix} .$$

Solution:
Form the characteristic polynomial:

$$\begin{aligned} P(r) &:= \det(\mathbf{A} - r\mathbf{I}) = \det \begin{pmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{pmatrix} \\ &= (-r)\left((-r)(-r) - 1\right) - (-r-1) + (1+r) = -r^3 + 3r + 2 \\ &= (r-2)(r+1)^2 \end{aligned}$$

The eigenvectors are found from the definition:

$$\mathbf{A}\mathbf{v}_i = r_i \mathbf{v}_i \ :$$

or

$$\begin{pmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

1. Eigenvalue $r_1 = 2$:

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Here we solve the system as followes: start with the homogeneous system:

$$\begin{array}{rcrcrcl} -2x & + & y & + & z & = & 0 \, , \\ x & + & -2y & + & z & = & 0 \, , \\ x & + & y & + & -2z & = & 0 \, . \end{array}$$

Multiply the third row time 2 and add to the first row
subtract the third row from the second row:

$$\begin{array}{rcrcrcl} & + & 3y & - & 3z & = & 0 \, , \\ & - & 3y & + & 3z & = & 0 \, , \\ x & + & y & - & 2z & = & 0 \, . \end{array}$$

The first two equations are now redundant; we set $z = 1$ and find

$$y = 1 \ , \ x = 2z - y = 1 \ .$$

Any multiple of this vector is also an eigenvector, i.e.

$$\mathbf{v}_1 = s \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

for s arbitrary is also an eigenvector of eigenvalue $r = 2$.

2. Eigenvalues $r_2 = r_3 = -1$ (here we have a case where two of the roots of the characteristic polynomial are equal (not yet done in class, but I will demonstrate the process of finding the eigenvectors anyway):

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \ , \ \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \ .$$

Here we have only one equation for the three unknowns x, y, z and we can therefore find two independent vectors that satisfy it

$$x + y + z = 0 \Rightarrow x = 1 \ , \ y = -1 \ , \ z = 0 \ ,$$

or

$$x + y + z = 0 \Rightarrow x = 1 \ , \ y = 0 \ , \ z = -1 \ .$$

Not all problems with equal eigenvalues can lead to as many eigenvectors as the multiplicity of the eigenvalue (there are "defective" matrices!).

7 Problem 9.5.11

Find the general solution for the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ for the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & \frac{3}{4} \\ -5 & 3 \end{pmatrix} .$$

Solution:

8 Problem 9.5.17

Consider the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, $t \geq 0$, with

$$\mathbf{A} = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} .$$

Solution:

1. Show that the matrix $\mathbf{A}$ has eigenvalues $r_1 = 2$ and $r_2 = -2$ with corresponding eigenvectors

$$\mathbf{u}_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} , \ \mathbf{u}_2 = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} .$$

Form the characteristic polynomial:

$$\begin{aligned} P(r) &:= \det(\mathbf{A} - r\mathbf{I}) = \det \begin{pmatrix} 1-r & \sqrt{3} \\ \sqrt{3} & -1-r \end{pmatrix} \\ &= (r^2 - 1) - 3 = r^2 - 4 = (r-2)(r+2) = 0 \end{aligned}$$

The eigenvectors are found from the definition:

$$\mathbf{A}\mathbf{u}_i = r_i \mathbf{u}_i \ :$$

or

$$\begin{pmatrix} 1-r & \sqrt{3} \\ \sqrt{3} & -1-r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

which gives a solution in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1+r \\ \sqrt{3} \end{pmatrix} .$$

(a) Eigenvalue $r_1 = 2$:
The solution is

$$\mathbf{u}_1 = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} .$$

(b) Eigenvalue $r_2 = -2$:
The solution is

$$\mathbf{u}_2 = \begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix} .$$

Having found the two eigenvectors we can now write the general solution of the system as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{2t} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix} ,$$

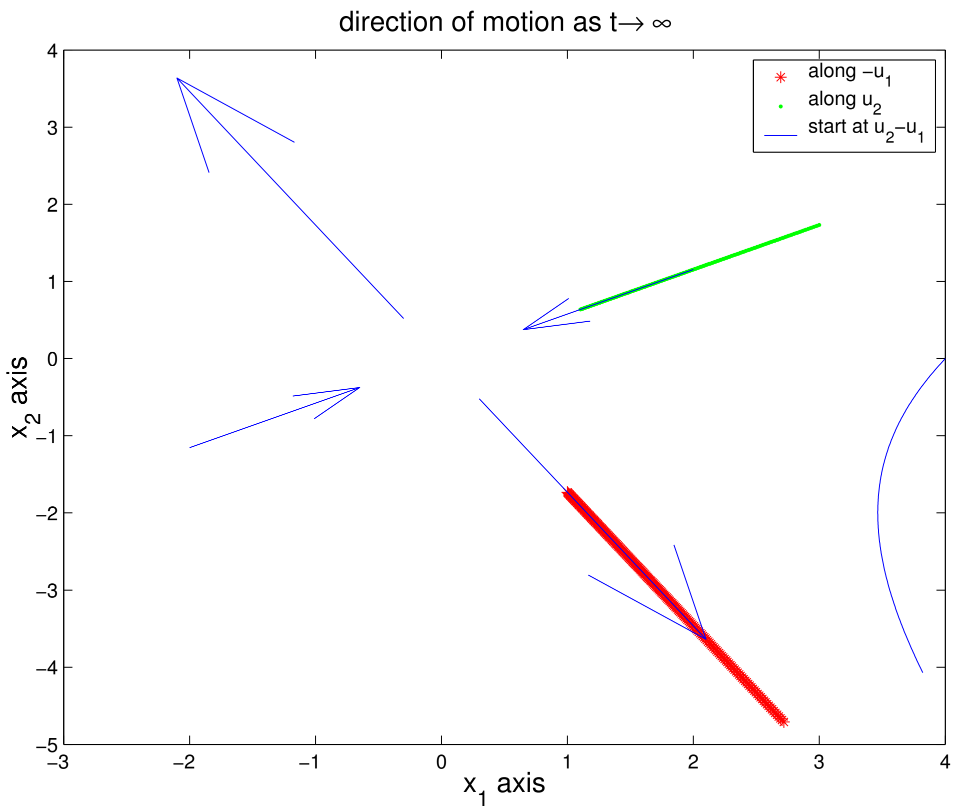
or

$$\mathbf{x}(t) = C_1 e^{-t}\mathbf{u}_1 + C_2 e^{-3t}\mathbf{u}_2 \ .$$

2. Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = -\mathbf{u}_1$.

3. Same as in (2) for initial vector $\mathbf{x}(0) = \mathbf{u}_2$.

4. Same as in (2) for initial vector $\mathbf{x}(0) = \mathbf{u}_2 - \mathbf{u}_1$.



9 Problem 9.5.20

Consider the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$, $t \geq 0$, with

$$\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} .$$

1. Show that the matrix $\mathbf{A}$ has eigenvalues $r_1 = -1$ and $r_2 = -3$ with corresponding eigenvectors

   $$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \; \mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

   Solution:
   Form the characteristic polynomial:

   $$\begin{aligned} P(r) &:= \det(\mathbf{A} - r\mathbf{I}) = \det \begin{pmatrix} -2-r & 1 \\ 1 & -2-r \end{pmatrix} \\ &= (2+r)^2 - 1 = r^2 + 4r + 3 = (r+1)(r+3) = 0 \end{aligned}$$

   The eigenvectors are found from the definition:

   $$\mathbf{A}\mathbf{u}_i = r_i \mathbf{u}_i \; :$$

   or

   $$\begin{pmatrix} -2-r & 1 \\ 1 & -2-r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

   (a) Eigenvalue $r_1 = -1$:

   $$\begin{aligned} -x + y &= 0 , \\ x - y &= 0 . \end{aligned}$$

   The solution is

   $$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

   (b) Eigenvalue $r_2 = -3$:

   $$\begin{aligned} x + y &= 0 , \\ x + y &= 0 . \end{aligned}$$

   The solution is

   $$\mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

Having found the two eigenvectors we can now write the general solution of the system as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} ,$$

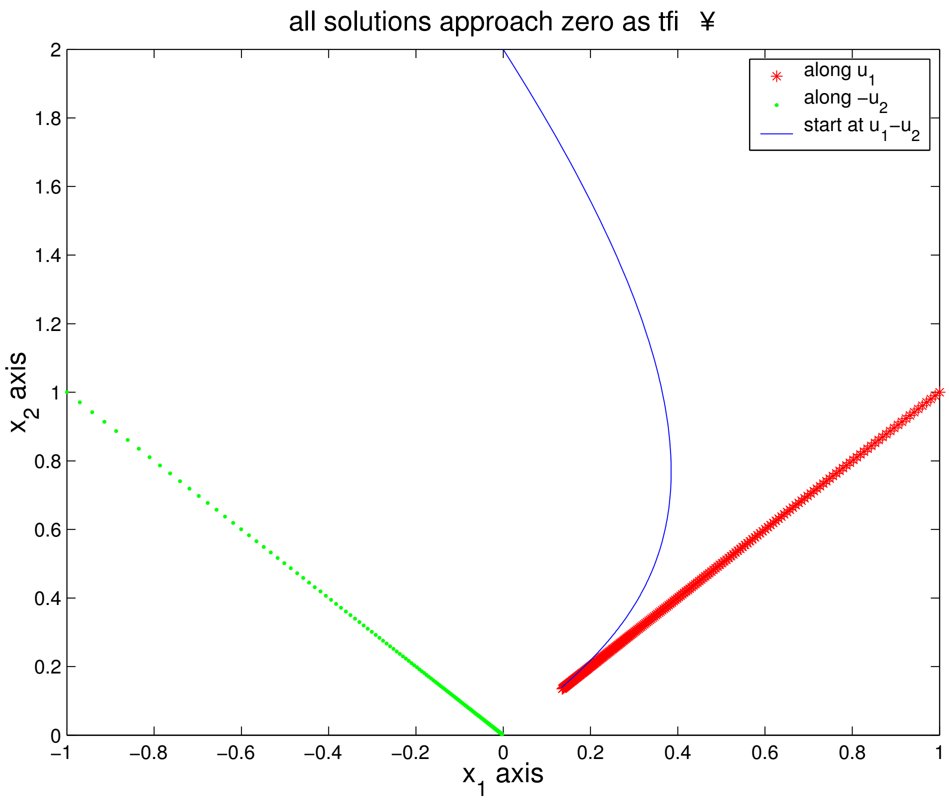
or

$$\mathbf{x}(t) = C_1 e^{-t}\mathbf{u}_1 + C_2 e^{-3t}\mathbf{u}_2 \ .$$

2. Sketch the trajectory of the solution having initial vector $\mathbf{x}(0) = \mathbf{u}_1$.

3. Same as in (2) for initial vector $\mathbf{x}(0) = -\mathbf{u}_2$.

4. Same as in (2) for initial vector $\mathbf{x}(0) = \mathbf{u}_1 - \mathbf{u}_2$.



MATLAB script

```
u1 = [1;1];
u2 = [1;-1];
t = linspace(0,2,201)
for i = 1:201
s1(:,i) = exp(-t(i))*u1(:);
s2(:,i) = -exp(-3*t(i))*u2(:);
end
s3 = s1 + s2;
plot(s1(1,:),s1(2,:),'r*',s2(1,:),s2(2,:),'g.',s3(1,:),s3(2,:),'b-')
xlabel('x_1 axis')
ylabel('x_2 axis')
title('plot of various trajectories; as time increases...
all solutions approach zero')
legend('along u_1','along -u_2','start at u_1-u_2')
```