Solutions, 316-XX

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7.9¡1*,2,3,4,11,15,19*,20,22*,23*,24¿ CAUTION: there may be errors!!!

1 Problem 7.9.1

Solve the IVP using Laplace transforms

$$\begin{aligned} x' &= 3x - 2y, \quad x(0) = 1 \\ y' &= 3y - 2x, \quad x(0) = 1 \end{aligned}$$

Solution:

$$\begin{aligned} \mathcal{L}\left\{x' = 3x - 2y\right\} &\Rightarrow \quad sX - 3X + 2Y = 1 \\ \mathcal{L}\left\{y' = 3y - 2x\right\} &\Rightarrow \quad sY - 3Y + 2X = 1 \end{aligned}$$

or

$$\begin{aligned} (s-3)X + 2Y = 1 \quad &, \quad 2X + (s-3)Y = 1 \\ ((s-3)^2 - 4)X = (s-3) - 2 \quad &, \quad Y = \frac{1}{2}\left(1 - (s-3)X\right) \\ X = \frac{(s-3)-2}{(s-3)^2-4} \quad &, \quad Y = \frac{1}{2}\left(1 - (s-3)\frac{s-5}{(s-3)^2-4}\right) \\ X = \frac{(s-3)-2}{(s-3)^2-4} \quad &, \quad Y = \frac{(s-3)-2}{(s-3)^2-4} \end{aligned}$$

giving

$$x(t) = y(t) = e^{3t}\left(\cosh 2t - \sinh 2t\right) \ .$$

2 Problem 7.9.19

Solve the IVP using Laplace transforms

$$\begin{aligned} x' &= 3x + y - 2z, \quad & x(0) = -6 \\ y' &= -x + 2y + x, \quad & y(0) = 2 \\ z' &= 4x + y - 3x, \quad & z(0) = -12 \end{aligned}$$

Solution:

$$\begin{aligned} \mathcal{L}\{x' = 3x + y - 2z\} &\Rightarrow sX - 3X - Y + 2Z = -6 \\ \mathcal{L}\{y' = -x + 2y + z\} &\Rightarrow sY + X - 2Y - Z = 2 \\ \mathcal{L}\{z' = 4x + y - 3x\} &\Rightarrow sZ - 4X - Y + 3Z = -12 \end{aligned}$$

or

$$\begin{array}{rcrcrcr} (s-3)X & - & Y & + & 2Z & = & -6\ , \\ X & + & (s-2)Y & - & Z & = & 2\ , \\ -4X & - & Y & + & (s+3)Z & = & -12 \end{array}$$

giving (multiply 1st row by 1, subtract from the second row multiplied by $(s-3)$ and use the result in place of the second row; also multiply the first row my -4 and subtract from the third row multiplied by $(s-3)$, using the result in place of the third row. Both the second and third rows no longer contain X):

$$\begin{array}{rcrcrcr} (s-3)X & - & Y & + & 2Z & = & -6\ , \\ 0X & + & [(s-2)(s-3)+1]\,Y & - & [(s-3)+2]\,Z & = & 2(s-3)+6\ , \\ 0X & + & [-(s-3)-4]\,Y & + & [(s+3)(s-3)+8]\,Z & = & -12(s-3)-24 \end{array}$$

which is simplified to:

$$\begin{array}{rcrcrcr} (s-3)X & - & Y & + & 2Z & = & -6\ , \\ 0X & + & [(s-2)(s-3)+1]\,Y & - & (s-1)Z & = & 2s\ , \\ 0X & - & (s+1)Y & + & (s^2-1)Z & = & -12(s-1) \end{array}$$

Now multiply the second row by $s+1$ and add to the third row to eliminate Z; the system becomes:

$$\begin{array}{rcrcrcr} (s-3)X & - & Y & + & 2Z & = & -6\ , \\ 0X & + & [(s-2)(s-3)+1]\,Y & - & (s-1)Z & = & 2s\ , \\ 0X & + & (s+1)(s-2)(s-3)Y & + & 0Z & = & 2(s-3)(s-2) \end{array}$$

and solving:

$$
\begin{aligned}
Y &= \frac{2}{s+1} \\
Z &= -\frac{2s(s+1) - 2\left[(s-2)(s-3)+1\right]}{(s-1)(s+1)} \\
&= -\frac{12s-14}{(s-1)(s+1)} \\
X &= \frac{-6+Y-2Z}{s-3} \\
&= \frac{-6(s-1)(s+1)+2(s-1)+2(12s-14)}{(s-1)(s+1)(s-3)} \\
&= \frac{-6s^2+26s-24}{(s-1)(s+1)(s-3)} \\
&= \frac{(s-3)(-6s+8)}{(s-1)(s+1)(s-3)} = \frac{-6s+8}{(s-1)(s+1)}
\end{aligned}
$$

The partial fractions expansions are

$$
\begin{aligned}
Z &= -\frac{12s-14}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} \\
A &= -\frac{12s-14}{s+1}\bigg|_{s=1} = 1 \\
B &= -\frac{12s-14}{s-1}\bigg|_{s=-1} = -13 \Rightarrow \\
Z &= \frac{1}{s-1} - \frac{13}{s+1}
\end{aligned}
$$

and

$$
\begin{aligned}
X &= \frac{-6s+8}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} \\
A &= \frac{-6s+8}{s+1}\bigg|_{s=1} = 1 \\
B &= \frac{-6s+8}{s-1}\bigg|_{s=-1} = -7 \Rightarrow \\
X &= \frac{1}{s-1} - \frac{7}{s+1}
\end{aligned}
$$

so that we can now invert to find x, y, z:

$$\begin{aligned} x(t) &= e^t - 7e^{-t} \\ y(t) &= 2e^{-t} \\ z(t) &= e^t - 13e^{-t} \ . \end{aligned}$$

3 Problem 7.9.22

Given the coupled mass-spring system

$$\begin{aligned} m_1 \frac{d^2x}{dt^2} &= -k_1(x-y) \\ m_2 \frac{d^2y}{dt^2} &= k_1(x-y) - k_2 y \end{aligned}$$

with initial conditions:

$$y(0) = y'(0) = x'(0) = 0 \ , \ x(0) = -1 \ .$$

Solve using Laplace transforms, if $m_1 = 1$kg, $m_2 = 2$kg, $k_1 = 4$N/m, $k_2 = 10/3$ N/m.
Solution:

$$\begin{aligned} \mathcal{L}\left\{ m_1 \frac{d^2x}{dt^2} = -k_1(x-y) \right\} &\Rightarrow m_1(s^2X + s) = -k_1(X - Y) \\ \mathcal{L}\left\{ m_2 \frac{d^2y}{dt^2} = k_1(x-y) - k_2 y \right\} &\Rightarrow m_2 s^2 Y = k_1(X-Y) - k_2 Y \end{aligned}$$

or

$$\begin{aligned} (m_1 s^2 + k_1)X - k_1 Y &= -m_1 s \ , \\ -k_1 X + (m_2 s^2 + k_1 + k_2)Y &= 0 \end{aligned}$$

giving

$$\begin{pmatrix} (m_1 s^2 + k_1) & -k_1 \\ -k_1 & (m_2 s^2 + k_1 + k_2) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -m_1 s \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{-m_1 s}{m_1 m_2 s^4 + [m_1(k_1+k_2)+m_2 k_1)]\, s^2 + k_1 k_2} \begin{pmatrix} m_2 s^2 + k_1 + k_2 \\ k_1 \end{pmatrix}$$

Substituting the numerical values of the quantities involved we get:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{-s}{2s^4 + [(4+10/3)+8)]\, s^2 + 40/3} \begin{pmatrix} 2s^2 + 4 + 10/3 \\ 4 \end{pmatrix}$$

or

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{-s}{2s^4 + (46/3)s^2 + 40/3} \begin{pmatrix} 2s^2 + 22/3 \\ 4 \end{pmatrix}$$

which simplifies as

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{-s}{2(s^2+20/3)(s^2+1)} \begin{pmatrix} 2s^2 + 22/3 \\ 4 \end{pmatrix}$$

so, finally we have the partial fractions expansions:

$$\begin{aligned} Y(s) &= \frac{-2s}{(s^2+20/3)(s^2+1)} \\ &= -2s\left(\frac{A}{s^2+20/3} + \frac{B}{s^2+1}\right) \end{aligned}$$

which is satisfied if we find A,B for

$$\begin{aligned} \frac{1}{(s^2+20/3)(s^2+1)} &= \frac{A}{s^2+20/3} + \frac{B}{s^2+1} \\ A &= \lim_{z\to -20/3} \frac{1}{z+1} = -3/17 \\ B &= \lim_{z\to -1} \frac{1}{z+20/3} = 3/17 \Rightarrow \\ Y(s) &= -2s\left(\frac{-3/17}{s^2+20/3} + \frac{3/17}{s^2+1}\right) = \frac{(6/17)s}{s^2+20/3} - \frac{(6/17)s}{s^2+1} \Rightarrow \\ y(t) &= \frac{6}{17}\cos\sqrt{\frac{20}{3}}t - \frac{6}{17}\cos t \end{aligned}$$

where we used $z = s^2$ to speed up the reduction. Also, since

$$\begin{aligned}
X(s) &= \frac{-s(2s^2 + 22/3)}{2(s^2 + 20/3)(s^2 + 1)} \\
&= \frac{-2s(s^2 + 20/3) + 6s}{2(s^2 + 20/3)(s^2 + 1)} \\
&= -\frac{s}{s^2 + 1} + 3s\frac{1}{(s^2 + 20/3)(s^2 + 1)} \\
&= -\frac{s}{s^2 + 1} + 3s\left(\frac{-3/17}{s^2 + 20/3} + \frac{3/17}{s^2 + 1}\right) \\
X(s) &= \frac{-8/17}{s^2 + 1} - \frac{9/17}{s^2 + 20/3} \Rightarrow \\
x(t) &= -\frac{8}{17}\cos 1 - \frac{9}{17}\cos\sqrt{\frac{20}{3}}t
\end{aligned}$$

4 Problem 7.9.23

The RLC network has equations (as derived in class):

$$\begin{aligned} L_1\frac{dI_1}{dt} + R_1 I_1 + R_2(I_1 - I_3) &= E \\ L_3\frac{dI_3}{dt} - R_2(I_1 - I_3) &= 0 \end{aligned}$$

where

$$R_1 = 2\Omega \ , \ R_2 = 1\Omega \ , \ E = 6V \ , \ L_3 = 0.1H \ , \ L_1 = 0.2H \ ,$$

and

$$I_2 = I_1 - I_3 \ .$$

Solve for the currents assuming $I_1(0) = I_2(0) = I_3(0) = 0$.
Solution:

$$\mathcal{L}\left\{L_1\frac{dI_1}{dt} + R_1 I_1 + R_2(I_1 - I_3) = E\right\} \ \Rightarrow \ L_1 s J_1 + R_1 J_1 + R_2(J_1 - J_3) = \frac{E}{s}$$

$$\mathcal{L}\left\{L_3\frac{dI_3}{dt} - R_2(I_1 - I_3) = 0\right\} \ \Rightarrow \ L_3 s J_3 - R_2(J_1 - J_3) = 0$$

or, rearranging:

$$\begin{aligned} \left(L_1 s + R_1 + R_2\right) J_1 - R_2 J_3 &= \frac{E}{s} \\ -R_2 J_1 + \left(L_3 s + R_2\right) J_3 &= 0 \end{aligned}$$

or, substituting numerical values:

$$\begin{aligned} \left(0.2s + 3\right) J_1 - J_3 &= \frac{6}{s} \\ -J_1 + \left(0.1s + 1\right) J_3 &= 0 \end{aligned}$$

and using matrix notation:

$$\begin{pmatrix} 0.2s + 3 & -1 \\ -1 & 0.1s + 1 \end{pmatrix} \begin{pmatrix} J_1 \\ J_3 \end{pmatrix} = \begin{pmatrix} \frac{6}{s} \\ 0 \end{pmatrix}$$

so that

$$
\begin{aligned}
\begin{pmatrix} J_1 \\ J_3 \end{pmatrix} &= \begin{pmatrix} 0.2s+3 & -1 \\ -1 & 0.1s+1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{6}{s} \\ 0 \end{pmatrix} \\
&= \frac{1}{(0.2s+3)\,(0.1s+1)-1} \begin{pmatrix} 0.1s+1 & 1 \\ 1 & 0.2s+3 \end{pmatrix} \begin{pmatrix} \frac{6}{s} \\ 0 \end{pmatrix} \\
\begin{pmatrix} J_1 \\ J_3 \end{pmatrix} &= \frac{6}{s(0.02s^2+0.5s+2)} \begin{pmatrix} 0.1s+1 \\ 1 \end{pmatrix} \\
&= \frac{6}{s((1/50)s^2+(1/2)s+2)} \begin{pmatrix} (1/10)s+1 \\ 1 \end{pmatrix} \\
&= \frac{300}{s(s^2+25s+100)} \begin{pmatrix} (1/10)s+1 \\ 1 \end{pmatrix} \\
&= \left(\frac{1}{100}\frac{1}{s} - \frac{1}{75}\frac{1}{s+5} + \frac{1}{300}\frac{1}{s+20}\right) \begin{pmatrix} 30s+300 \\ 300 \end{pmatrix} \Rightarrow \\
\begin{pmatrix} J_1 \\ J_3 \end{pmatrix} &= \begin{pmatrix} \frac{30s+300}{100}\frac{1}{s} - \frac{30(s+5)+150}{75}\frac{1}{s+5} + \frac{30(s+20)-300}{300}\frac{1}{s+20} \\ 3\frac{1}{s} - 4\frac{1}{s+5} + \frac{1}{s+20} \end{pmatrix} \\
&= \begin{pmatrix} .3 + 3\frac{1}{s} - \frac{30}{75} - 2\frac{1}{s+5} + .1 - \frac{1}{s+20} \\ 3\frac{1}{s} - 4\frac{1}{s+5} + \frac{1}{s+20} \end{pmatrix} \\
&= \begin{pmatrix} 3\frac{1}{s} - 2\frac{1}{s+5} - \frac{1}{s+20} \\ 3\frac{1}{s} - 4\frac{1}{s+5} + \frac{1}{s+20} \end{pmatrix} \Rightarrow \\
\begin{pmatrix} I_1 \\ I_3 \end{pmatrix} &= \begin{pmatrix} 3 - 2e^{-5t} - e^{-20t} \\ 3 - 4e^{-5t} + e^{-20t} \end{pmatrix} .
\end{aligned}
$$

Then

$$I_2 = I_1 - I_3 = 2e^{-5t} - 2e^{-20t} .$$