Solutions, 316-XVIII

April 9, 2003

Notation:

$$u(t-c) \equiv u_c(t) := \begin{cases} 0 & , \quad t \leq c \\ 1 & , \quad c < t \end{cases}$$

7.6¡1,2,3*,4,5*,6,15*,16,29,30,31*,35*,36¿
CAUTION: there may be errors!!!

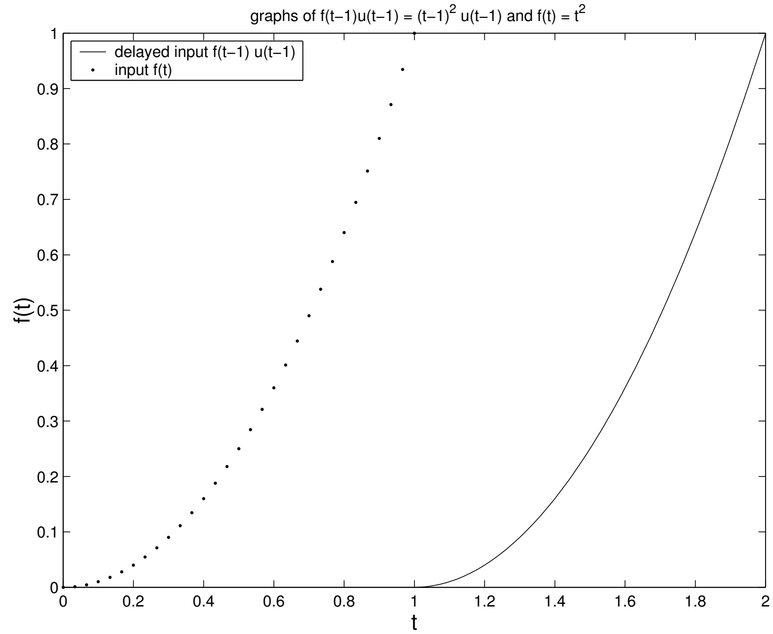
1 Problem 7.6.1

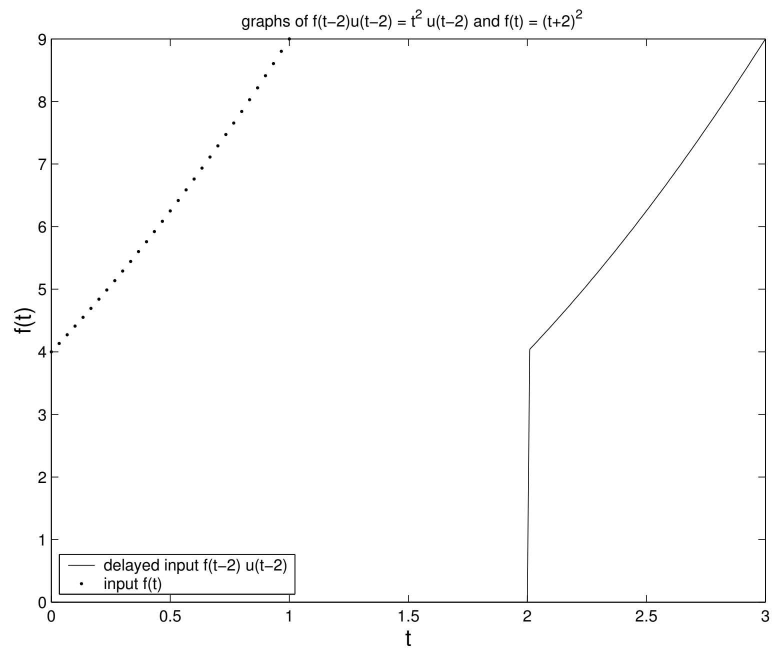
Sketch the graph and determine the Laplace transform.

$$(t-1)^2 u(t-1) \ .$$

Solution:

$$\begin{aligned} \mathcal{L}\left\{t^2\right\} &= \frac{2}{s^3} \rightarrow \\ \mathcal{L}\left\{(t-1)^2 u(t-1)\right\} &= e^{-s}\frac{2}{s^3} \end{aligned}$$


graphs of f(t−1)u(t−1) = (t−1)^2 u(t−1) and f(t) = t^2
delayed input f(t−1) u(t−1)
input f(t)
f(t)
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


graphs of f(t−2)u(t−2) = t^2 u(t−2) and f(t) = (t+2)^2
delayed input f(t−2) u(t−2)
input f(t)
f(t)
t
0
1
2
3
4
5
6
7
8
9
0
0.5
1
1.5
2
2.5
3

2 Problem 7.6.3

Sketch the graph and determine the Laplace transform.

$$t^2 u(t-2)\ .$$

Solution:
(hint: $t = (t-2) + 2$).
We have

$$f(t-2) = t^2 = [(t-2)+2]^2 \Rightarrow f(t) = (t+2)^2 = t^2 + 4t + 4\ .$$

Then

$$\begin{aligned} \mathcal{L}\left\{t^2 + 4t + 4\right\} &= \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \rightarrow \\ \mathcal{L}\left\{\left((t-2)^2 + 4(t-2) + 4\right) u(t-2)\right\} &= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right) \end{aligned}$$

3 Problem 7.6.4

Sketch the graph and determine the Laplace transform.

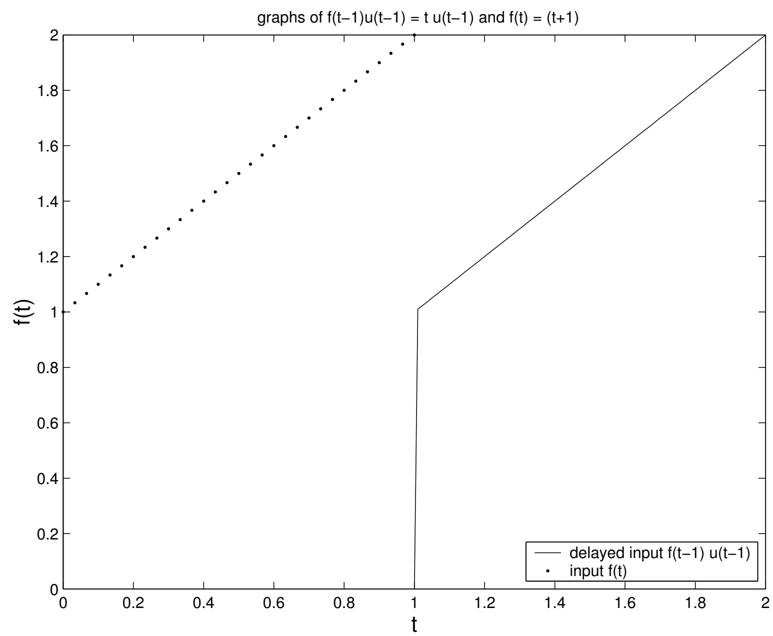
$$tu(t-1)\ .$$

Solution:
(hint: $t = (t-1)+1$ is required so that the function is converted to the canonical form $f(t-a)u(t-a)$).
Since

$$tu(t-1) = \left[(t-1)+1\right]u(t-1)$$

we have

$$\begin{aligned} \mathcal{L}\left\{t+1\right\} &= \frac{1+s}{s^2} \rightarrow \\ \mathcal{L}\left\{\left[(t-1)+1\right]u(t-1)\right\} &= e^{-s}\frac{1+s}{s^2} \end{aligned}$$



4 Problem 7.6.5

Express the given function using unit step functions and compute Laplace transform.

$$g(t) = \begin{cases} 0 & , \quad 0 \le t \le 1 \\ 2 & , \quad 1 < t \le 2 \\ 1 & , \quad 2 < t \le 3 \\ 3 & , \quad 3 \le t \end{cases}$$

Solution:

$$\begin{aligned} g(t) &= 2u(t-1) - u(t-2) + 2u(t-3) \Rightarrow \\ G(s) &= \frac{1}{s}\left(2e^{-s} - e^{-2s} + 2e^{-3s}\right) \ . \end{aligned}$$

5 Problem 7.6.5

Express the given function using unit step functions and compute Laplace transform.

$$g(t) = \begin{cases} 0 & , \quad 0 \le t \le 2 \\ t+1 & , \quad 2 \le t \end{cases}$$

Solution :

$$\begin{aligned} g(t) &= \begin{cases} 0 & , \quad 0 \le t \le 2 \\ t+1 & , \quad 2 \le t \end{cases} \rightarrow \\ g(t) &= (t+1)u(t-2) = [(t-2)+3]\, u(t-2) \Rightarrow \\ \mathcal{L}\{t+3\} &= \frac{1+3s}{s^2} \rightarrow \\ \mathcal{L}\{[(t-2)+3]\, u(t-2)\} &= e^{-2s}\frac{1+3s}{s^2} \end{aligned}$$

6 Problem 7.7.15

Determine an inverse Laplace transform for the function

$$\frac{se^{-3s}}{s^2+4s+5} .$$

Solution:

$$\begin{aligned}
\frac{s}{s^2+4s+5} &= \frac{(s+2)-1}{(s+2)^2+1} , \\
\mathcal{L}^{-1}\left\{\frac{s-1}{s^2+1}\right\} &= \cos t - \sin t \Rightarrow \\
\mathcal{L}^{-1}\left\{\frac{(s+2)-1}{(s+2)^2+1}\right\} &= e^{-2t}\left(\cos t - \sin t\right) \Rightarrow \\
\mathcal{L}^{-1}\left\{\frac{se^{-3s}}{s^2+4s+5}\right\} &= e^{-2(t-3)}\left[\cos\left(t-3\right) - \sin\left(t-3\right)\right] u(t-3)
\end{aligned}$$

7 Problem 7.7.16

Determine an inverse Laplace transform for the function

$$\frac{e^{-s}}{s^2+4} .$$

Solution:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} &= \sin 2t \Rightarrow \\
\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+4}\right\} &= \sin 2(t-1)\ u(t-1)
\end{aligned}$$

8 Problem 7.7.30

Solve the given IVP with Laplace transforms; graph the solution.

$$w'' + w = u(t-2) - u(t-4) \; ; \; w(0) = 1 \; , \; w'(0) = 0 \; .$$

Solution:

$$\begin{aligned}
w'' + w &= u(t-2) - u(t-4) \; ; \; w(0) = 1 \; , \; w'(0) = 0 \\
\mathcal{L}\left\{w'' + w\right\} &= \mathcal{L}\left\{u(t-2) - u(t-4)\right\} \\
\left(s^2 + 1\right) W(s) - s &= \frac{e^{-2s} - e^{-4s}}{s} \\
W(s) &= \frac{s}{1+s^2} + \frac{e^{-2s} - e^{-4s}}{s} \frac{1}{1+s^2}
\end{aligned}$$

where for the last expression we need

$$\begin{aligned}
\frac{1}{s(1+s^2)} &= \frac{A}{s} + \frac{Bs + C}{1+s^2} \\
A &= \lim_{s \to 0} s \frac{1}{s(1+s^2)} = 1 \\
\frac{Bs + C}{1+s^2} &= \frac{1}{s(1+s^2)} - \frac{1}{s} \\
&= \frac{1 - (s^2 + 1)}{s(1+s^2)} = -\frac{s}{1+s^2} \\
\frac{1}{s(1+s^2)} &= \frac{1}{s} - \frac{s}{1+s^2}
\end{aligned}$$

so that

$$\begin{aligned}
W(s) &= \frac{s}{1+s^2} + \left(e^{-2s} - e^{-4s}\right)\left(\frac{1}{s} - \frac{s}{1+s^2}\right) \\
y(t) &= \cos t + (1 - \cos(t-2))\, u(t-2) - (1 - \cos(t-4))\, u(t-4)
\end{aligned}$$

9 Problem 7.7.31

Solve the given IVP with Laplace transforms; graph the solution.

$$y'' + y = t - (t-4)u(t-2) \; ; \; y(0) = 0 \; , \; y'(0) = 1 \; .$$

Solution:

$$\begin{aligned}
y'' + y &= t - (t-4)u(t-2) \; ; \; y(0) = 0 \; , \; y'(0) = 1 \\
\mathcal{L}\{y'' + y\} &= \mathcal{L}\{t - [(t-2) - 2]\, u(t-2)\} \\
\left(s^2+1\right) Y(s) - 1 &= \frac{s - e^{-2s}(1-2s)}{s^2} \\
Y(s) &= \frac{1}{1+s^2} + \frac{s - e^{-2s}(1-2s)}{s^2} \frac{1}{1+s^2}
\end{aligned}$$

Now

$$\begin{aligned}
\frac{1}{s(s^2+1)} &= \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow A = \lim_{s\to 0} \frac{1}{s^2+1} = 1 \\
\frac{Bs+C}{s^2+1} &= \frac{1}{s(s^2+1)} - \frac{1}{s} = \frac{1-(s^2+1)}{s(s^2+1)} = -\frac{s}{s^2+1} \Rightarrow \\
\frac{1}{s(s^2+1)} &= \frac{1}{s} - \frac{s}{s^2+1}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{s^2(s^2+1)} &= \frac{1}{z(z+1)} = \frac{1}{2}\frac{1}{z} - \frac{1}{2}\frac{1}{z+1} \Rightarrow \\
\frac{1}{s^2(s^2+1)} &= \frac{1}{2}\frac{1}{s^2} - \frac{1}{2}\frac{1}{s^2+1}
\end{aligned}$$

Combining, we rewrite $Y(s)$ as

$$\begin{aligned}
Y(s) &= \frac{1}{1+s^2} + \frac{1+2e^{-2s}}{s(s^2+1)} - \frac{e^{-2s}}{s^2(s^2+1)} \Rightarrow \\
&= \frac{1}{1+s^2} + \left(1+2e^{-2s}\right)\left(\frac{1}{s} - \frac{s}{s^2+1}\right) - e^{-2s}\left(\frac{1}{2}\frac{1}{s^2} - \frac{1}{2}\frac{1}{s^2+1}\right)
\end{aligned}$$

resulting in

$$y(t) = \cos t + 1 - \sin t + u(t-2)\left(2 + 2\sin(t-2) - \frac{1}{2}(t-2) + \frac{1}{2}\cos(t-2)\right)$$

10 Problem 7.7.35

Solve the IVP using Laplace transforms.

$$z'' + 3z' + 2z = e^{-3t}u(t-2) \; ; \; z(0) = 2 \; , \; z'(0) = -3 \; .$$

Solution:

$$\begin{aligned}
\mathcal{L}\{z'' + 3z' + 2z\} &= \mathcal{L}\left\{e^{-3(t-2)-6}u(t-2)\right\} \\
\left(s^2 + 3s + 2\right) Z(s) - 2s + 3 - 6 &= \frac{e^{-6}}{s+3}e^{-2s} \Rightarrow \\
Z(s) = \frac{2s+3}{s^2+3s+2} + e^{-6}e^{-2s}\frac{1}{(s+3)(s^2+3s+2)} \; .
\end{aligned}$$

Since $s^2 + 3s + 2 = (s+1)(s+2)$ we have

$$\begin{aligned}
\frac{1}{(s+3)(s+2)(s+1)} &= \frac{A}{s+3} + \frac{B}{s+2} + \frac{C}{s+1} \Rightarrow \\
A &= \lim_{s\to -3} \frac{1}{(s+1)(s+2)} = \frac{1}{2} \\
B &= \lim_{s\to -2} \frac{1}{(s+1)(s+3)} = -1 \\
C &= \lim_{s\to -1} \frac{1}{(s+2)(s+3)} = \frac{1}{2} \Rightarrow \\
\frac{1}{(s+1)(s+2)(s+3)} &= \frac{1}{2}\frac{1}{s+3} - \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+1} \; ,
\end{aligned}$$

$$\begin{aligned}
\frac{2s+3}{(s+2)(s+1)} &= \frac{A}{s+1} + \frac{B}{s+2} \Rightarrow \\
A = \lim_{s\to -1} \frac{2s+3}{s+2} = 1 \quad &, \quad B = \lim_{s\to -2} \frac{2s+3}{s+1} = 1 \\
\frac{2s-1}{(s+2)(s+1)} &= \frac{1}{s+1} + \frac{1}{s+2}
\end{aligned}$$

Combining, we rewrite $Z(s)$ as

$$\begin{aligned}
Z(s) &= \frac{1}{s+2} + \frac{1}{s+1} + e^{-6}e^{-2s}\left(\frac{1}{2}\frac{1}{s+3} - \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+1}\right) \Rightarrow \\
y(t) &= e^{-2t} + e^{-t} + e^{-6}u(t-2)\left(\frac{1}{2}e^{-3(t-2)} - e^{-2(t-2)} + \frac{1}{2}e^{-(t-2)}\right)
\end{aligned}$$

	t-domain $(f(t))$	s-domain $(F(s))$
1	$f(t)$	$F(s)$
2	$C_1 f_1(t) + C_2 f_2(t)$	$C_1 F_1(s) + C_2 F_2(s)$
3	1	$\frac{1}{s}$
4	t	$\frac{1}{s^2}$
5	t^n	$\frac{n!}{s^{n+1}}$
6	e^{at}	$\frac{1}{s-a}$
7	$e^{at} f(t)$	$F(s-a)$
8	$\cos bt$	$\frac{s}{s^2+b^2}$
9	$\sin bt$	$\frac{b}{s^2+b^2}$
10	$f'(t)$	$sF(s) - f(0)$
11	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
12	$tf(t)$	$-F'(s)$
13	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
14	$\int_0^t g(\tau) h(t-\tau) d\tau$	$G(s)H(s)$
15	$\int_0^t g(\tau) d\tau$	$\frac{1}{s} G(s)$
16	$f(t-a)u(t-a)$	$e^{-as} F(s)$

Table 1: *Useful Laplace transforms*