Solutions, 316-XII

March 13, 2003

Unchecked solutions; still need to verify. Please let me know if you find any typos!

1 Problem 4.12.1

Sketch the frequency response curve for the system in which $m = 4$, $k = 1$, $b = 2$.

Solution:

Given the general forced-damped oscillator system

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F_0 \cos \gamma t$$

the particular solution, y_p is given by

$$y_p(t) = \frac{F_0}{\sqrt{(k - m\gamma)^2 + b^2\gamma^2}} \sin(\gamma t + \theta) \ ,$$

where

$$\tan \theta = \frac{k - m\gamma^2}{b\gamma} \ .$$

The factor

$$M(\gamma) := \frac{1}{\sqrt{(k - m\gamma)^2 + b^2\gamma^2}}$$

describes the amplitude of the responce relative to that of the input. The resonance curve is its graph:

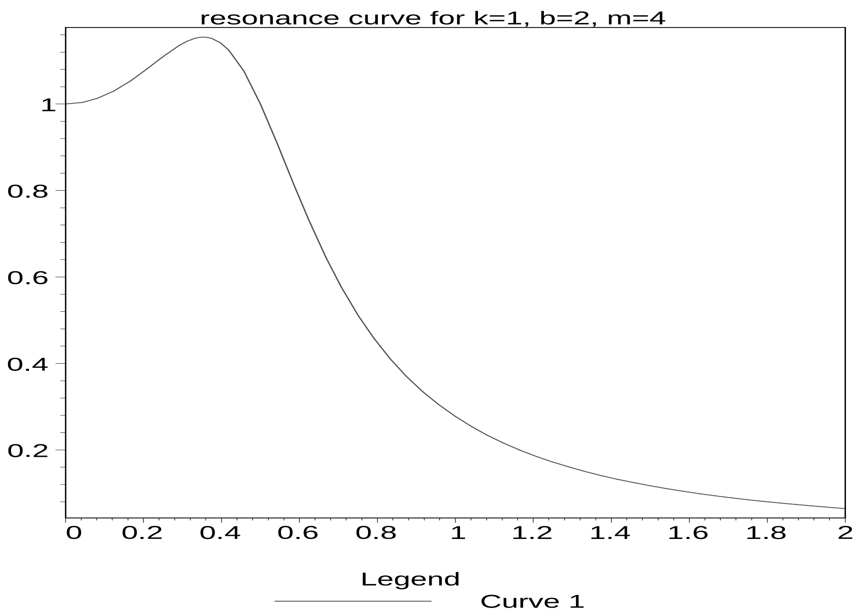
```
> restart;
> res := gamma -> 1/(sqrt((k-m*gamma^2)^2+(b*gamma)^2));
```

$$res := \gamma \to \frac{1}{\sqrt{(k - m\,\gamma^2)^2 + b^2\,\gamma^2}}$$

```
> res1 := gamma -> 1/(sqrt((1-4*gamma^2)^2+(2*gamma)^2));
```

$$res1 := \gamma \to \frac{1}{\sqrt{(1 - 4\,\gamma^2)^2 + 4\,\gamma^2}}$$

```
> plot(res1,0..2,axes=BOXED,title="resonance curve for k=1, b=2,
> m=4");
```



2 Problem 4.12.4

Determine the equation of motion and sketch the solution for an undamped system at resonance governed by:

$$\begin{aligned} \frac{d^2y}{dt^2} + y &= 5\cos t \ ; \\ y(0) = 0 \quad &; \quad y'(0) = 0 \ . \end{aligned}$$

Solution:
Consider the undamped forced oscillator:

$$\begin{aligned} \frac{d^2y}{dt^2} + y &= 5\cos \gamma t \ ; \\ y(0) = 0 \quad &; \quad y'(0) = 0 \ . \end{aligned}$$

In the next problem the solution for this equation with general coefficients is derived; it is

$$y(t;\gamma) = F_0 \frac{\cos\gamma t - \cos\omega t}{k - m\gamma^2} \ .$$

Here we set $F_0 = 5$, $m = 1$, $k = 1$ so that $\omega = \sqrt{(k/m)} = 1$ so that

$$y(t;\gamma) = 5\frac{\cos\gamma t - \cos t}{1 - \gamma^2}$$

and if we take the limit $\gamma \to 1$ we are led to:

$$\begin{aligned} y(t) &= \lim_{\gamma\to 1} 5\frac{\cos\gamma t - \cos t}{1-\gamma^2} \\ &= 5\lim_{\gamma\to 1} \frac{\frac{d}{d\gamma}\left(\cos\gamma t - \cos t\right)}{\frac{d}{d\gamma}\left(1-\gamma^2\right)} \\ &= 5\lim_{\gamma\to 1} \frac{-t\sin\gamma t}{-2\gamma} \\ &= \frac{5}{2} t \sin t \end{aligned}$$

where we used l'Hospital's rule to take the limit since both numerator and denominator approached zero. This is the alternative derivation of the particular solution in the case of resonance discussed in class. Now, the general solution is

$$y_g(t) = C_1 \cos t + C_2 \sin t + \frac{5}{2} t \sin t$$

and substituting for the Initial Conditions:

$$y_g(0) = C_1 = 0\ ,\ y_g'(0) = C_2 = 1$$

so, finally:

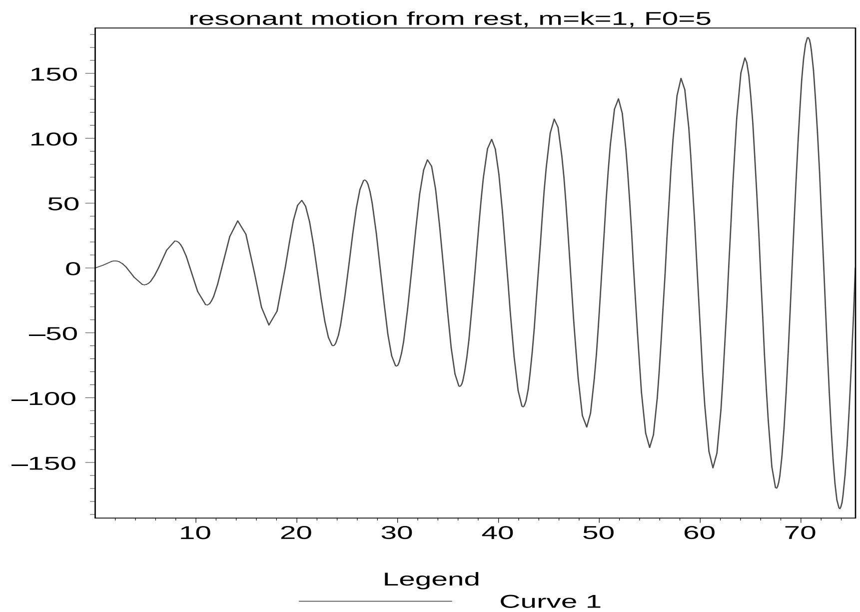
$$y(t) = \sin t + \frac{5}{2} t \sin t\ .$$

This is plotted below.

```
> restart;
> yg := t -> sin(t)+(5/2)*t*sin(t);
```

$$yg := t \to \sin(t) + \frac{5}{2}\, t \sin(t)$$

```
> plot(yg,0..24*Pi,axes=BOXED,title="resonant motion from rest,
m=k=1,
> F0=5");
```



3 Problem 4.12.5

An undamped system is governed by

$$\begin{aligned} m\frac{d^2y}{dt^2} + ky &= F_0 \cos \gamma t \\ y(0) = 0 \quad &; \quad y'(0) = 0 \ . \end{aligned}$$

where $\gamma \neq \omega := \sqrt{k/m}$.

1. Find the equation of motion of the system.
   Solution:
   Consider a particular solution in the form

   $$y_p = A \cos \gamma t + B \sin \gamma t$$

   Substituting into the equation and grouping similar terms together we have

   $$(-mA\gamma^2 + kA) \cos \gamma t + (-mB\gamma^2 + kB) \sin \gamma t = F_0 \cos \gamma t$$

   so that

   $$A = \frac{F_0}{k - m\gamma^2} \ ; \ B = 0 \ .$$

   The general solution is then

   $$y_g(t) = C_1 \cos \omega t + C_2 \sin \omega t + \frac{F_0}{k - m\gamma^2} \cos \gamma t \ .$$

   Satisfying the initial conditions we find:

   $$\begin{aligned} y_g(0) = C_1 + \frac{F_0}{k - m\gamma^2} &= 0 \\ y_g'(0) = C_2 \omega &= 0 \end{aligned}$$

   so that the solution to the IVP is given by:

   $$y(t) = F_0 \frac{\cos \gamma t - \cos \omega t}{k - m\gamma^2} \ .$$

2. Use trigonometric identities to show that the solution can be written in the form

$$y(t) = \frac{2F_0}{m\left(\omega^2 - \gamma^2\right)} \sin\left(\frac{\omega + \gamma}{2} t\right) \sin\left(\frac{\omega - \gamma}{2} t\right) .$$

Solution:
Using the trigonometric identity:

$$\begin{aligned}
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
2\cos A \cos B &= \cos(A - B) + \cos(A + B) \\
2\sin A \sin B &= \cos(A - B) - \cos(A + B) \\
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
2\sin A \cos B &= \sin(A - B) + \sin(A + B) \\
2\cos A \sin B &= \sin(A + B) - \sin(A - B)
\end{aligned}$$

so that, if we set

$$A = \frac{a+b}{2} \ , \ B = \frac{a-b}{2}$$

i.e.

$$a = A + B \ , \ b = A - B$$

we have

$$\cos b - \cos a = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

Applying these identities to our problem:

$$\cos\gamma t - \cos\omega t = 2\sin\left(\frac{\omega + \gamma}{2} t\right) \sin\left(\frac{\omega - \gamma}{2} t\right) .$$

This leads to the desired answer.

3. When γ is near ω, then $\omega - \gamma$ is small, while $\omega + \gamma$ is relatively large compared with $\omega - \gamma$. Hence $y(t)$ can be viewed as the product of a slowly varying function, $\sin[(\omega - \gamma/2)t]$, and a rapidly varying function, $\sin[(\omega + \gamma/2)t]$. The net effect is a sine function $y(t)$ with frequency $(\omega - \gamma/2)/4\pi$ which serves as the time-amplitude of a sine function with frequency $(\omega + \gamma/2)/4\pi$. This vibration phenomenon is referred to as **beats** and is used in tuning stringed instruments. This same phenomenon in electronics is called **amplitude modulation**. To

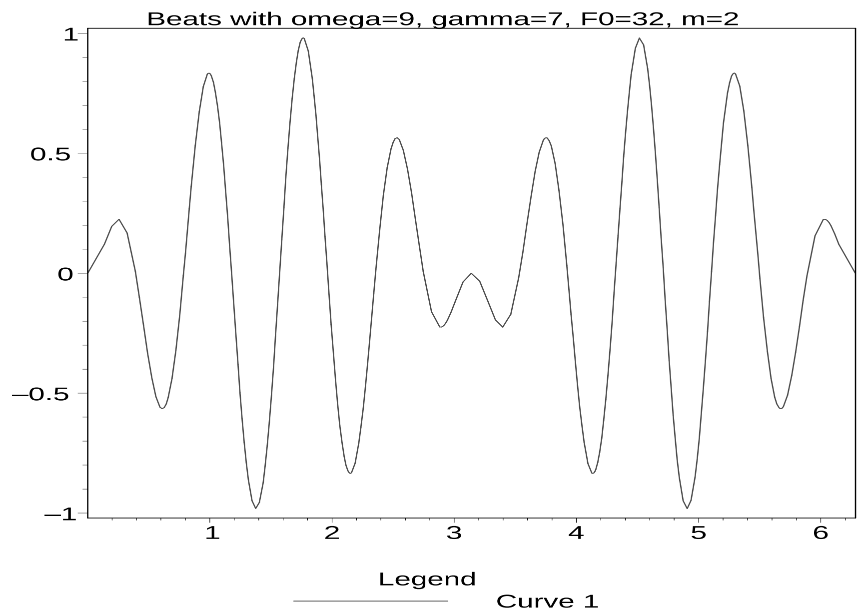
illustrate this phenomenon sketch the curve $y(t)$ for $F_0 = 32$, $m = 2$, $\omega = 9$, and $\gamma = 7$.

Solution:

```
> restart;
> expr := t -> (2*32/(2*(9^2-7^2)))*sin((9+7)*t/2)*sin((9-7)*t/2);
```

$$expr := t \to \sin(8\,t)\sin(t)$$

```
> plot(expr,0..2*Pi,axes=BOXED,title="Beats with omega=9, gamma=7,
> F0=32, m=2");
```



4 Problem 4.12.8

The responce of an overdamped system to a constant force is governed by the equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F_0 \cos \gamma t$$

with $m = 2$, $b = 8$, $k = 6$, $F_0 = 18$, and $\gamma = 0$. If the system starts from rest ($y(0) = y'(0) = 0$), compute and sketch the displacement $y(t)$. What is the limiting value of $y(t)$ as $t \to +\infty$? Interpret this physically.

Solution:

The general solution to the forced-damped system is

$$y_g(t) = C_1 e^{\mu t} \cos \nu t + C_2 e^{\mu t} \sin \nu t + R \sin(\gamma t + \theta)$$

with

$$\mu = -\frac{b}{2m} = -2 \ , \ \nu = \frac{\sqrt{4mk - b^2}}{2m} = 1 \ , \ R = \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}} \ ,$$

and

$$\tan \theta = \frac{k - m\gamma^2}{b\gamma} \ .$$

For the case where $\gamma = 0$, i.e. a constant force, the solution is

$$y_g(t) = C_1 e^{\mu t} \cos \nu t + C_2 e^{\mu t} \sin \nu t + \frac{F_0}{k}$$

and substituting for the initial conditions:

$$y_g(0) = C_1 + \frac{F_0}{k} = 0 \Rightarrow C_1 = -\frac{18}{6} = -3 \ ,$$

$$y_g'(0) = C_1\mu + C_2\nu = -2C_1 + C_2 = 0 \to C_2 = -6$$

Finally:

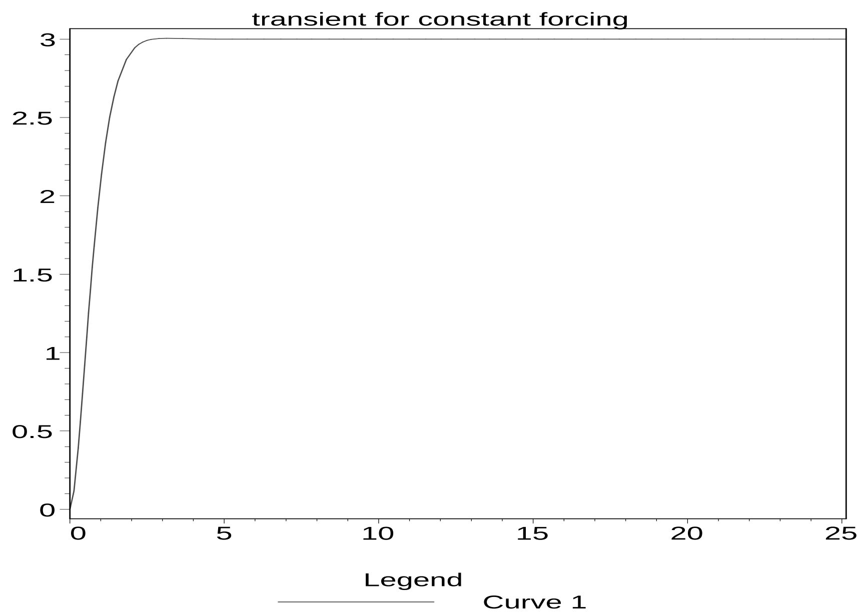
$$y(t) = -3e^{-2t} \cos t - 6e^{-2t} \sin t + 3 \ .$$

The solution asymptotes to the constant value 3 as $t \to \infty$.

```
> restart;
> yp := t -> exp(-2*t)*(-3*cos(t) - 6*sin(t)) + 3;
```

$$yp := t \to e^{(-2\,t)}\,(-3\cos(t) - 6\sin(t)) + 3$$

```
> plot(yp,0..8*Pi,axes=BOXED,title="transient for constant
> forcing");
```


transient for constant forcing
3
2.5
2
1.5
1
0.5
0
0
5
10
15
20
25
Legend
Curve 1

5 Problem 4.12.12

A 2-kg mass is attached to a spring hanging from the ceiling, thereby causing the spring to stretch 20 cm upon coming to rest at equilibrium. At time $t = 0$, the mass is displaced 5 cm below the equilibrium position and released. At this same instant, an external force $F(t) = 0.3\cos t$ N is applied to the system. If the damping constant of the system is 5 N-sec/m, determine the equation of motion for the mass. What is the resonance frequency for the system?

Solution:

The spring constant is found from the displacement as:

$$k = mg/d = \frac{2kg * 9.81m}{sec^2 * 0.05m} = 392.4N/m \ .$$

Substituting in the general expressions given in the previous problem, we have:

$$\mu = -\frac{b}{2m} = -1.25 \ , \ \nu = \frac{\sqrt{4mk - b^2}}{2m} = 13.864 \ ,$$

$$R = \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}} = \frac{.3}{\sqrt{(392.4 - 2)^2 + 5^2}} = .000768 \ ,$$

and

$$\tan\theta = \frac{k - m\gamma^2}{b\gamma} = \frac{392.4 - 2}{5} = 78.08 \rightarrow \theta = 89.27$$

so that

$$y(t) = e^{-1.25t}\left(C_1 \cos 13.864t + C_2 \sin 13.864t\right) + .000768 \sin\left(t + 89.27\right)$$

and satisfying the initial conditions

$$y(0) = 0 = C_1 + .000768 \sin 89.27 \rightarrow C_1 = -.00074 \ ,$$

$$y'(0) = -1.25C_1 + 13.864C_2 + .000768 \cos 89.27 = 0 \rightarrow C_2 = -.000081$$

so, finally

$$y(t) = 10^{-4}\left(e^{-1.25t}\left(-7.4 \cos 13.864t - .81 \sin 13.864t\right) + 7.68 \sin\left(t + 89.27\right)\right)$$

The resonant frequency is equal to the pseudofrequency or

$$\nu = 13.864 \ .$$

This is of course the displacement from the new equilibrium; to find the total displacement, add .05m to these values.