Solutions, 316-X

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1 Problem 6.3.15

Find the differential operator that annihilates $e^{2x} - 6e^x$.
Solution:
In general, the operator $D - r$ annihilates the exponential e^{rx} so the desired operator is $(D-2)(D-1) = D^2 - 3D + 2$. Then

$$\left(D^2 - 3D + 2\right)\left(e^{2x} - 6e^x\right) = (D-1)\left[(D-2)e^{2x}\right] - 6(D-2)\left[(D-1)e^x\right] = 0+0 = 0$$

2 Problem 6.3.16

Find the differential operator that annihilates $x^2 - e^x$.
Solution:
The operator $D - 1$ annihilates the exponential e^x while the operator D^3 annihilates x^2 so that

$$D^3(D-1)\left[x^2 - e^x\right] = (D-1)\left[D^3 x^2\right] - D^3\left[(D-1)e^x\right] = 0$$

3 Problem 6.3.22

Use the annihilator method to determine the form of a particular solution to the DE

$$y'' + 6y' + 8y = \left(D^2 + 6D + 8\right) y e^{3x} - \sin x \ .$$

Solution:
The characteristic equation is $r^2 + 6r + 8 = 0$ with roots $r = -2, -4$. Since

$\pm i$ and 3 are not roots, try

$$y_p(t) = A\cos x + B\sin x + Ce^{3x} \ .$$

To discover this with the annihilator method:
$(D-3)$ annihilates e^{3x}
D^2+1 annihilates $\sin x$
So we have:

$$(D-3)\left(D^2+1\right)(D+2)(D+4)\,y = (D-3)\left(D^2+1\right)\left(e^{3x} - \sin x\right) = 0$$

So y_g contains terms that are annihilated by each of these operators. Then

$$y_g = C_1 e^{-2x} + C_2 e^{-4x} + Ae^{3x} + B\cos x + C\sin x \ .$$

And to find the particular solution set $C_1 = C_2 = 0$.

4 Problem 6.3.24

Use the annihilator method to determine the form of a particular solution to the DE

$$\theta'' - \theta = (D^2-1)\theta = xe^x \ .$$

Solution:
The characteristic equation is $r^2 - 1 = 0$, with roots ± 1. Since 1 is not a root, the solution has the form

$$y_p(t) = (Ax^2 + Bx)e^x \ .$$

(We dont include the term Ae^x since it is a multiple of the homogeneous solution. To discover this with the annihilator method:
$(D-1)$ annihilates e^x
$(D-1)^2$ annihilates xe^x
$(D-1)^3$ annihilates x^2e^x
Since $(D^2-1) = (D+1)(D-1)$ we have:

$$(D-1)^2(D^2-1)y = (D+1)(D-1)^3 y = (D-1)^2\left(xe^x\right) = 0 \ .$$

Then, the most general function annihilated by the operator $(D+1)(D-1)^3$ is

$$y_g = C_1 e^x + C_2 e^{-x} + \left(Ax^2 + Bx\right)e^x \ .$$

Again, set $C_1 = C_2 = 0$ to find particular solution.

5 Problem 6.3.27

Use the annihilator method to determine the form of a particular solution to the DE

$$\left(D^2 + 2D + 2\right) = e^{-x}\cos x + x^2 \ .$$

Solution:

$$D^2 + 2D + 2 = (D+1)^2 + 1$$

so that the characteristic equation, $(r+1)^2 + 1 = 0$ has roots $r = -1 \pm i$. The homogeneous solution here is

$$y_h = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x \ .$$

Now, the operator $D^2 + 2D + 2$ annihilates $e^{-x}\cos x$
while the operator D^3 annihilates x^2. Thus

$$\begin{aligned} D^3\left(D^2+2D+2\right)^2 y &= D^3\left(D^2+2D+2\right)\left(e^{-x}\cos x + x^2\right) \\ &= D^3\left[\left(D^2+2D+2\right)e^{-x}\cos x\right] + \left(D^2+2D+2\right)\left[D^3 x^2\right] \\ &= 0 \end{aligned}$$

So that the general solution is annihilated by the operator

$$D^3\left(D^2+2D+2\right)^2$$

i.e.

$$y_g = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x + Axe^{-x}\cos x + Bxe^{-x}\sin x + (Cx^2 + Dx + E)$$

and to find particular solution set $C_1 = C_2 = 0$.