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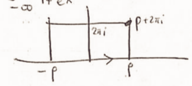
METHODS OF C-INTEGRATION

Bnd:

$$\frac{\pi}{\sin a\pi} = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx, \; 0<a<1$$

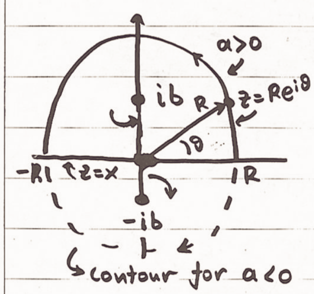


(1) $\int_{-\infty}^{\infty} f(x)\,dx$; $|f(z)| < \frac{M}{|z|^{1+\delta}}$ for $|z|>R_0$

continuation of $f(x)$ on either upper or lower half plane

Ex. $I(a) = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2}\,dx$; $a, b > 0$

$$I = \mathrm{Re}\, I_1 = \mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}\,dx = \int_{-\infty}^{\infty}\frac{e^{iax}}{x^2+b^2}\,dx \text{ (even)}$$



$$\int_{-R}^{R} \frac{e^{iax}}{x^2+b^2}\,dx + \int_0^{\pi} \frac{e^{iaz} R i e^{i\vartheta}\, d\vartheta}{R^2 e^{2i\vartheta}+b^2}$$

$I(a)$ as $R\to\infty$; $\to 0$

$$= 2\pi i \cdot \mathrm{Res}\,(ib) = 2\pi i \cdot \left.\frac{e^{iaz}}{z+ib}\right|_{ib} \qquad e^{iaz} = e^{-ab}, \; z+ib = 2ib$$

$$= \frac{\pi}{b} e^{-ab}$$

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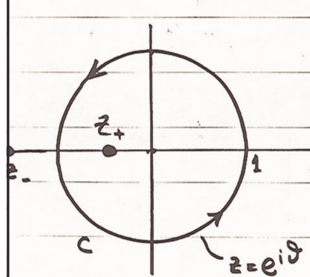
(2) $I = \int_0^{2\pi} R(\cos\vartheta, \sin\vartheta)\,d\vartheta = \oint_{C}^{2\pi} R(\cos\vartheta, \sin\vartheta)\,\frac{dz}{iz}$

$z = e^{i\vartheta}$; $dz = ie^{i\vartheta} d\vartheta = iz\,d\vartheta$, $\quad \frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)$

$I = 2\pi i \sum \mathrm{Res}\,(\tilde{R}(z))$ where

$$\tilde{R}(z) = R\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\cdot\frac{1}{z}$$

Ex: $I = \int_0^{2\pi} \frac{d\vartheta}{1+a\cos\vartheta}$; $|a|<1$



$$I = \frac{1}{i}\oint_{|z|=1} \frac{dz}{z}\,\frac{1}{1+\frac{a}{2}\left(z+\frac{1}{z}\right)} = \frac{2}{i}\oint_{|z|=1}\frac{dz}{az^2+2z+a}$$

$z_\pm = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2}-1}$; $z_+ z_- = 1$, $a \neq 1$

only one root inside (z_+)

$z_+ = -\frac{1}{a} + \sqrt{\frac{1}{a^2}-1}$

$$I = \frac{2}{i}\cdot 2\pi i \cdot \mathrm{Res}\,(z_+) = 4\pi\,\frac{1}{a(z_+ - z_-)} = \frac{2\pi}{\sqrt{1-a^2}}$$

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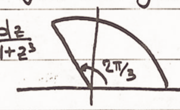
(3) $\int_0^\infty f(x)\,dx$ where $f(z)$ has symmetry

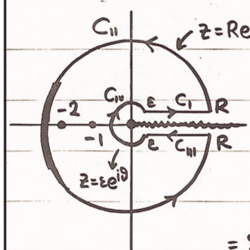
Ex: $\int_0^\infty \frac{dx}{1+x^3} \to \oint_c \frac{dz}{1+z^3}$



since $(re^{2\pi i/3})^3 = r^3$

This is not always possible!

Ex. $I = \int_0^\infty \frac{dx}{x^2+3x+2}$



Consider

$B = \oint_c \frac{\ln z\, dz}{z^2+3z+2}$ Show $C_{II}, C_{IV} \to 0$

Then $\int_{C_I} - \int_{C_{III}} = 2\pi i \sum \text{Res} = -2\pi i\, I$ (see next p.)

$$= 2\pi i \left(\frac{\ln(2e^{i\pi})}{\underset{z+1|_{z=-2}}{-2+1}} + \frac{\ln(e^{i\pi})}{\underset{z+2|_{z=-1}}{-1+2}} \right) = -2\pi i \ln 2$$

$\Rightarrow I = \ln 2$

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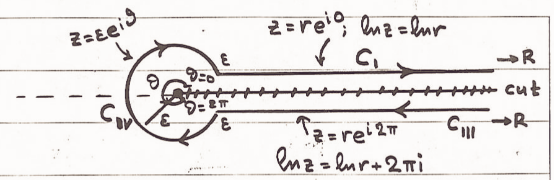

on C_I: $\frac{\ln z}{z^2+3z+2} = \frac{\ln r + 0i}{r^2+3r+2}$

on C_{III}: $\frac{\ln z}{z^2+3z+2} = \frac{\ln r + 2\pi i}{r^2+3r+2}$

$$\underset{\xrightarrow{(+)}}{\int_{C_I}} + \underset{\xleftarrow{(-)}}{\int_{C_{III}}} = \int_\varepsilon^R \frac{\ln r - (\ln r + 2\pi i)}{r^2+3r+2}\,dr = -2\pi i \int_\varepsilon^R \frac{dr}{r^2+3r+2} \xrightarrow[\varepsilon\to 0]{R\to\infty} -2\pi i\, I$$

C_{IV}: $z = \varepsilon e^{i\theta}$: $\left|\int_{C_{IV}} \cdots\right| \le \int_{C_{IV}} \frac{|\ln\varepsilon + i\theta||\varepsilon i e^{i\theta}|\,d\theta}{|\varepsilon^2 e^{2i\theta} + 2\varepsilon e^{i\theta} + 3|} \le \int_0^{2\pi} \frac{(|\ln\varepsilon| + |\theta|)\varepsilon\, d\theta}{3 - 2\varepsilon - \varepsilon^2} \xrightarrow{\varepsilon\to 0} 0$

C_{II}: $z = Re^{i\theta}$: $\left|\int_{C_{II}} \cdots\right| \le \int \frac{(|\ln R| + |\theta|)R\, d\theta}{R^2 - 3R - 2} \xrightarrow{R\to\infty} 0$

$\varepsilon \ln\varepsilon \xrightarrow{\varepsilon\to 0} 0$, $\frac{\ln R}{R} \xrightarrow{R\to\infty} 0$

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In general, if $|f(z)| < \frac{M}{R^{1+\delta}}$, $\overset{R}{\overset{\parallel}{|z|}} > R_0$

then can evaluate

$\int_0^\infty f(x)dx$ by key-hole contour for

$\oint_c^\infty \ln z\, f(z)\, dz$.

Note: above we assumed that $f(z)$ (the continuation of $f(x)$ for complex values) had no poles on the (+) real axis. In fact we can also allow simple poles on the real axis. For that we need to introduce the

"Cauchy Principal Value"

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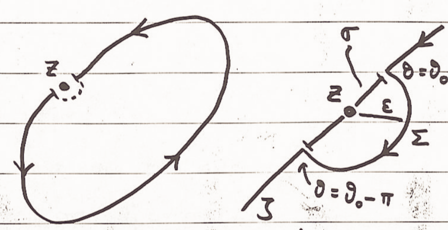

$$I = \oint_c \frac{f(\zeta)}{\zeta - z} d\zeta = \lim_{\varepsilon \to 0} \int_{C-\sigma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Cauchy principal value, let "lips" of excluded interval close at the same rate.

let $C' = C - \sigma$; then

$\oint_{C'+\Sigma} \frac{f(\zeta)}{\zeta - z} d\zeta = 0$ (no poles inside).

And: $\lim_{\varepsilon \to 0} \oint_{C'} \equiv$ P.V. $\oint_c$ (desired integral)

But $\lim_{\varepsilon \to 0} \int_\Sigma \frac{f(\zeta)}{z - \zeta} d\zeta = f(z) \lim_{\varepsilon \to 0} \int_{\vartheta_0}^{\vartheta_0 - \pi} \frac{i\cancel{\varepsilon e^{i\vartheta}} d\vartheta}{\cancel{\varepsilon e^{i\vartheta}}} = -\pi i f(z)$

$\zeta = z + \varepsilon e^{i\vartheta}$, $\vartheta_0 - \pi \le \vartheta \le \vartheta_0$ $\quad f(\zeta) = f(z + \varepsilon e^{i\vartheta}) \xrightarrow[\varepsilon \to 0]{} f(z)$

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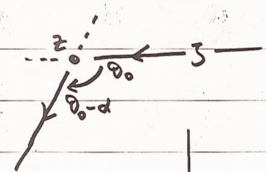
So: $PV\oint \frac{f(\zeta)}{\zeta - z}\,d\zeta = \pi i f(z)$

(instead of $2\pi i f(z)$ if z was interior)

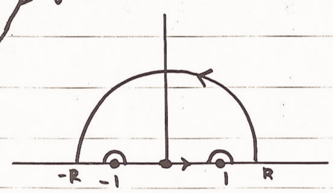
<u>Note:</u> if contour had an angle α at z (i.e. it was not smooth), then

$$PV\oint \frac{f(\zeta)}{\zeta - z}\,d\zeta = i\alpha f(z)$$



<u>Ex</u>: $PV\int_{-\infty}^{\infty} \frac{dx}{x^2-1} = \pi i \left(\frac{1}{2}\frac{1}{z+1}\Big|_{z=1} + \frac{1}{2}\frac{1}{z-1}\Big|_{z=-1} \right)$

$= 0$



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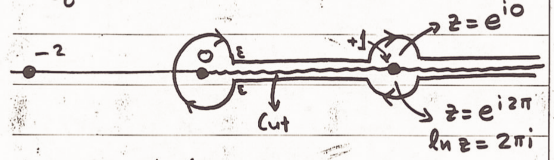
Ex: (simple pole on the branch cut)

$I = \int_0^\infty \frac{dz}{z^2+z-2}$: $\oint \frac{\ln z\, dz}{z^2+z-2}$ $\ln z = 0$



Work as before. But now

$$-2\pi i I = 2\pi i \left(Res(z=-2) + \frac{1}{2} Res(z = e^{i0}) + \frac{1}{2} Res(e^{i2\pi}) \right)$$

note (+) sign

$Res(e^{i0}) = \frac{\ln z}{z+2}\Big|_{z=e^{i0}} = \frac{0}{3}$

$Res(e^{i2\pi}) = \frac{\ln z}{z+2}\Big|_{z=e^{i2\pi}} = \frac{2\pi i}{3}$

$Res(-2 = 2e^{i\pi}) = \frac{\ln z}{z-1}\Big|_{z=2e^{i\pi}} = \frac{\ln 2 + i\pi}{-3}$

$$I = -\left(\frac{\ln 2 + i\pi}{-3} + 0 + \frac{1}{2}\left(\frac{2\pi i}{3}\right) \right) = \ln 2 / 3$$