Math 563
Assignment 4, due Thursday, November 1

Exercises 2-5 below do not use the normed space structure of $L^p(A)$, so for any $0 < p < \infty$ regard $L^p(A)$ to be the space of measurable functions f defined on a measurable set A such that $\int_A |f|^p < \infty$ and $\|f\|_p = (\int_A |f|^p)^{1/p}$. In particular, 2 and 3 do not require that $p \geq 1$.

1. Stein & Shakarchi, Chapter 2, Exercise #17, p. 93

2. Prove that if $0 < p < q < r \leq \infty$, then $L^q(A) \subset L^p(A) + L^r(A)$; that is, each $f \in L^q(A)$ can be written as $f = g + h$, the sum of a function in $g \in L^p(A)$ and a function in $h \in L^r(A)$.

3. Suppose $m(A) < \infty$ and $0 < p < q \leq \infty$. Show that $L^q(A) \subset L^p(A)$ and
$$\|f\|_p \leq \|f\|_q \, (m(A))^{\frac{1}{p} - \frac{1}{q}}.$$

4. (Generalized Hölder inequality) Suppose that
$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} = \frac{1}{r}$$
with $1 \leq p_j \leq \infty$ for $j = 1, \ldots, k$ and $1 \leq r \leq \infty$. If $f_j \in L^{p_j}(A)$ for $j = 1, \ldots, k$, then $\Pi_{j=1}^k f_j \in L^r(A)$ and
$$\|\Pi_{j=1}^k f_j\|_r \leq \Pi_{j=1}^k \|f_j\|_{p_j}$$
(Hint: using induction, reduce to the case where $k = 2$. Then derive this as a consequence of the usual Hölder inequality.)

5. (Convolution workout) Begin by working through parts (a), (b), (c) in Stein & Shakarchi, Chapter 2, Exercise #21 on your own (i.e. you do not need to hand them in). Then hand in the following problems:
   (a) Show that $\text{supp}(f * g) \subset \text{supp}(f) + \text{supp}(g)$.
   (b) Show that $f * g$ is integrable on $\mathbb{R}^d$ whenever f and g are integrable on $\mathbb{R}^d$ and that
   $$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$
   (c) Show that $f * g$ is uniformly continuous when f is integrable and g is bounded.
   (d) Show that if f and g are both integrable on $\mathbb{R}^d$, then $(f * g)(x) \to 0$ as $|x| \to \infty$. (Hint: by part (a), this would be clear if $\text{supp}(f)$ and $\text{supp}(g)$ were bounded. While you can't assume this, integrable functions "almost" have bounded support.)