Math 563, Fall 2016
Assignment 5, due Wednesday, October 26

Hand in the following exercises. Note that there is a difference between an "exercise" and a "problem" in the text, below we refer to the former.

1. Stein-Shakarchi, Exercise #2, Chapter 6.
2. Stein-Shakarchi, Exercise #3, Chapter 6.
3. Stein-Shakarchi, Exercise #32, Chapter 1. (Yes, go back to Chapter 1!)
4. Given σ-algebras $\mathcal{M}$ and $\mathcal{N}$ defined as collections of subsets in spaces X, Y, respectively. A function $f : X \to Y$ is said to be $(\mathcal{M}, \mathcal{N})$-*measurable* if $f^{-1}(E) \in \mathcal{M}$ whenever $E \in \mathcal{N}$. If X, Y are metric spaces, then f is said to be *Borel measurable* if it is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable (with $\mathcal{B}_X, \mathcal{B}_Y$ denoting the Borel sets in the respective spaces).
   (a) Suppose $\mathcal{E}$ is a collection of sets in Y and that $\mathcal{N}$ is the smallest σ-algebra containing $\mathcal{E}$. Show that f is $(\mathcal{M}, \mathcal{N})$ measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for every $E \in \mathcal{E}$. Hint: show that
   $$\widetilde{\mathcal{N}} = \{E \subset Y : f^{-1}(E) \in \mathcal{M}\}$$
   is a σ-algebra.
   (b) Show that any continuous function $f : X \to Y$ is Borel measurable.
5. Let $f : [0,1] \to [0,1]$ be the Cantor-Lebesgue function, and define $g : [0,1] \to [0,2]$ by $g(x) = x + f(x)$.
   (a) Prove that g is a bijection and that $g^{-1} : [0,2] \to [0,1]$ is continuous. Note: you can make quick work of the surjection proof and continuity of the inverse by appealing to topological arguments and Theorem 4.17 in Rudin respectively.
   (b) Prove that the Cantor set $\mathcal{C}$ satisfies $m(g(\mathcal{C})) = 1$.
   (c) By a previous exercise, $g(\mathcal{C})$ contains a Lebesgue nonmeasurable set A. Prove that $g^{-1}(A)$ is Lebesgue measurable but not Borel.
   (d) There exists a Lebesgue measurable function F and a continuous function G such that $F \circ G$ is not Lebesgue measurable.

Reading: Stein and Shakarchi, continue through Chapter 6.

On your own: Stein-Shakarchi, Exercise #1, Chapter 6.